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On the change of amplitude of interacting solitary waves

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In this paper the solitary-wave solutions of the Euler equations of motion are treated as a perturbation of the KdV equation. We show, analytically, that the amplitudes of two solitary waves are altered after interaction. This change in amplitude is calculated, showing that the smaller wave decreases in amplitude and the larger wave increases in amplitude.

1. Introduction

The discovery of the solitary wave was made by Scott-Russell in 1834 on the Edinburgh to Glasgow Union Canal. Later, in his report to the British Association Scott-Russell (1844) named this phenomenon 'the great wave of elevation'. It was formed after a barge, that was being towed by a team of horses, hit an obstacle in the canal and came to a halt. On a bridge over the canal near this spot a plaque to commemorate this discovery was erected during the *Soliton 82* conference at Heriot Watt University. At the unveiling ceremony a team of participants at the conference towed a barge in an unsuccessful attempt to recreate the phenomenon of the solitary wave. This failure was attributed to lack of power, a fact borne out by experiments of Huang *et al.* (1982) and Ertekin (1984) (see also Ertekin, Webster & Wehausen 1984, 1986). These experiments show that only when the Froude number, derived from the speed of the boat, is greater than about 0.7, are visible solitary waves produced. The name solitary wave was given to Scott-Russell's discovery by later authors because it consisted of a single humped wave.

Analytic approximations for the solitary wave were given by, for example, Boussinesq (1872) and Korteweg & de Vries (1895). The latter authors derived an equation, now called the KdV equation, that is a model for the unidirectional or one-dimensional propagation of long waves of small amplitude. One set of solutions of this equation is a family of steady solitary waves. The name soliton was given to the solitary-wave solution of the KdV equation following the discovery that the result of the nonlinear interaction of a pair of unequal solitary waves leaves them unaltered, except for a phase shift. In 1977, Miles extended this result in his study of solitary waves propagating in different directions. We shall refer to this as two-dimensional propagation of solitary waves. Miles had two categories of interaction: weak and strong. Weak interaction takes place when the angle between the normals to the direction of propagation is not small. The interaction is termed weak because the first approximation to the solution is just a superposition of the two waves. The interaction term first appears at second order, in the form of an additional term and a phase shift. This pattern is repeated at higher orders, so that the nth-order travelling-wave solution completely determines the $(n+1)$th order interaction term.

While the convergence of this approximation scheme has not been demonstrated there seemed no evidence to suppose that it cannot be continued to any required order, at least for small amplitudes, until the third-order theory of Su & Mirie (1980, 1982) appeared. They demonstrated that a dispersive tail appeared at that order but found no evidence of change of amplitude. This may of course be consistent at third order but cannot be entirely correct since the dispersive tail has to take some energy from the travelling waves, and hence their amplitudes must decrease. The strong-interaction case, which occurs when the angle between the normals is small, is different. Miles obtained the solution and showed that it is closely related to the KdV solution for two unidirectional waves. His result shows that, as in the one-dimensional case, the waves are unaltered in shape and in direction of propagation, but suffer a phase shift. However, in both the one- and two-dimensional strong interaction, the solution to second order is difficult to obtain, and it is not clear that the pattern is repeated at this order.

Johnson (1982) attempts to find the solution to the problem of the oblique interaction of a large and a small solitary wave. As in the theory of Miles there are strong and weak interactions. Johnson's solution for the weak interaction appears to be complete and is consistent with the pattern outlined above for the small-amplitude case. However, the theory relies on the assumption that the two waves remain unaltered in amplitude after interaction. His strong-interaction theory is less complete and the full solution is not obtained. (He has to assume the existence of a function that he is unable to determine.) What now appears to be a fundamental weakness of his theory is the assumption that the large wave is unaltered in shape during the interaction. Fenton & Rienecker (1982) provide a Fourier-series method for solving the full Euler equations. Their method assumes periodic waves and is applicable to waves travelling in the same direction or in opposite directions. The application to solitary waves involved waves of very long, but none-the-less, finite wavelength. They present only one interaction for waves going in the same direction, and that is for two waves whose initial amplitudes were 0.1035 and 0.3252. After the interaction they claim that the faster wave had increased in amplitude by 0.0036 and that the slower wave had decreased in magnitude by 0.0018. They also claim that the total energy is unchanged, and that no dispersive tail is produced. However this cannot be entirely correct. If the waves have genuinely separated into two solitary waves then the above result has to mean that the total energy has increased. This is because the energy of solitary waves as a function of amplitude, given by Longuet-Higgins & Fenton (1974, figure 5) indicates that $\mathrm{d}E/\mathrm{d}a > 0$ for all amplitudes less than about 0.6. That no dispersive tail is produced is also inconsistent with the fact that there is a change of amplitude. This is because of the time reversibility of the solution. The Fenton & Rienecker method for waves travelling in opposite directions shows that both waves decrease in amplitude as a result of interaction, a result consistent with an energy transfer to the dispersive tail. Again, however, the claim is made that no dispersive tail is visible and that energy is conserved.

In this paper we present theoretical evidence that for strong interactions the amplitudes of both waves will be altered after interaction. We shall consider the case of unidirectional propagation only, and starting from the Euler equations for fluid flow we shall derive the interaction equations correct to second order. These equations represent a perturbation of the KdV equation and can be solved by perturbation methods developed by Karpman & Maslov (1977*a*, *b*, *c*), Keener & McLaughlin (1977), Kaup & Newell (1978), and used by Byatt-Smith (1987). The solution shows that there is a second-order change in the amplitudes of the interacting waves. This is still in agreement with Miles since his analysis, although nonlinear, is accurate to the same degree of approximation as the KdV equation, namely it provides a solution correct to first order. We should therefore refrain from using the word soliton to describe Scott-Russell's great wave of elevation, and refer to it simply as a solitary wave.

2. Perturbation equations

We follow the notation of Miles (1977) and use only dimensionless variables. Let (x, y) be horizontal and vertical coordinates, t time, $\alpha\eta$ the free-surface displacement, and ϕ the velocity potential. The variable y has been made dimensionless and stretched with respect to x so that the quiescent depth is one and all derivatives are of order one. The characteristic non-dimensional parameters that appear are α, the amplitude, and $1/\beta^{\frac{1}{2}}$, the wavelength. The boundary-value problem for inviscid irrotational motion is described by

$$\beta\phi_{xx} + \phi_{yy} = 0 \quad (0 < y < 1 + \alpha\eta), \tag{2.1}$$

$$\phi_y = 0 \quad (y = 0), \tag{2.2}$$

$$\eta_t + \alpha\phi_x\eta_x - \frac{\phi_y}{\beta} = 0 \quad (y = 1 + \alpha\eta), \tag{2.3}$$

and

$$\eta + \phi_t + \tfrac{1}{2}\alpha\phi_x^2 + \frac{\frac{1}{2}\alpha\phi_y^2}{\beta} = 0 \quad (y = 1 + \alpha\eta). \tag{2.4}$$

Following, for example, Miles (1977), we look for a solution in the form

$$\phi(x, y, t) = \sum_0^\infty \frac{(-\beta \mathrm{D}^2)^m\, g(x, t)\, y^{2m}}{(2m)!}, \tag{2.5}$$

where $\mathrm{D} \equiv \partial/\partial x$.

Eliminating η between (2.3) and (2.4) yields

$$\eta = -g_t + \tfrac{1}{2}\beta g_{xxt} - \tfrac{1}{2}\alpha g_x^2 - \frac{1}{4!}\beta^2 g_{xxxxt} + \tfrac{1}{2}\alpha\beta\{g_x g_{xxx} - 2g_{xxt}\, g_t - g_{xx}^2\} + O(\alpha^3), \tag{2.6}$$

and

$$g_{xx} - g_{tt} = \tfrac{1}{6}\beta(3g_{xxtt} - g_{xxxx}) - \frac{\beta^2}{30} g_{xxxxxx} - \alpha\{(g_x)_t^2 + g_{xx}\, g_t\} + \alpha\beta\{\tfrac{1}{2}(g_x\, g_{xxx})_x - g_{xxx}\, g_{xx}\}$$
$$- \tfrac{1}{2}\alpha^2\{g_x(g_x)_x^2 + g_{xx}\, g_x^2\} + O(\alpha^3). \tag{2.7}$$

For cnoidal or solitary waves, β is proportional to α and it is customary to put $\beta = 3\alpha/4$. Then we introduce travelling coordinates, moving at the reference wave speed, given by

$$\xi = x - t, \quad \tau = \tfrac{1}{4}\alpha t, \tag{2.8}$$

the factor $\frac{1}{4}$ being introduced for convenience. Then with the notation $' \equiv \partial/\partial\xi$ we obtain

$$-\tfrac{1}{2}\alpha g_\tau - \alpha\{\tfrac{1}{4}g^{\mathrm{iv}} + \tfrac{3}{2}g'^2\} = -\alpha\{\tfrac{1}{16}g_{\tau\tau} + \tfrac{1}{4}g_\tau''' + \tfrac{1}{4}g'' g_\tau + g' g_\tau'\}$$
$$- \alpha^2\{\tfrac{1}{20}g^{\mathrm{vi}} + g' g^{\mathrm{iv}} + \tfrac{3}{4}g'' g''' + \tfrac{9}{2}g'' g'^2\} + O(\alpha^3). \tag{2.9}$$

To first order in α this reduces to

$$g_\tau' = -\tfrac{1}{2}g^{\mathrm{iv}} - 3g'^2, \tag{2.10}$$

so that

$$g_\tau = -\tfrac{1}{2}g''' - 3g'^2. \tag{2.11}$$

To reduce this to the KdV equation we introduce

$$u = -2g'. \tag{2.12}$$

Then to first order u satisfies the KdV equation

$$u_\tau = 3uu' - \tfrac{1}{2}u'''. \tag{2.13}$$

Using this approximation in the higher-order terms of (2.9) we obtain

$$u_\tau = 3uu' - \tfrac{1}{2}u''' + \alpha\{-\tfrac{19}{60}u^{\mathrm{iv}} + \tfrac{5}{8}uu'' + \tfrac{43}{32}u'^2\}', \tag{2.14}$$

where the term $O(\alpha^2)$ has been omitted.

This may be expressed as

$$u_\tau = \mathrm{X}^2(u) - \tfrac{19}{40}\alpha \mathrm{X}^3(u) + \frac{3\alpha}{16}\{-3uu''' + 5u'u'' + 19u^2u'\}, \tag{2.15}$$

where X^2 and X^3 are the first two operators in the hierarchy of KdV flows (see Byatt-Smith 1987 or McKean & van Morbeke 1975 for further details). These linear operators are given by

$$\mathrm{X}^2 = \mathrm{D}u + u\mathrm{D} - \tfrac{1}{2}\mathrm{D}^3, \tag{2.16}$$

and

$$\mathrm{X}^3 = \tfrac{15}{2}u\mathrm{D}u - 5(u''\mathrm{D} + \tfrac{1}{2}u''') + \tfrac{1}{4}\mathrm{D}^5, \tag{2.17}$$

where now $\mathrm{D} \equiv \partial/\partial\xi$.

3. Solitary wave

The solitary wave is by hypothesis a function of the single variable

$$\theta = \kappa(x - ct) = \kappa(\xi - (c-1)t) = \kappa\left(\frac{\xi - 4(c-1)\tau}{\alpha}\right). \tag{3.1}$$

If we look for a solution of the form

$$u = G(\theta), \quad \kappa = \kappa_1 + \kappa_2\alpha \ldots, \quad c = 1 + c_1\alpha + c_2\alpha^2 \ldots, \tag{3.2}$$

then the solution appropriate for a wave whose amplitude is α_n/α is

$$G = G_0 + \tfrac{1}{2}\alpha G_0^2 + \tfrac{1}{4}\alpha_n F_0, \tag{3.3}$$

$$\kappa_n = \left(\frac{\alpha_n}{\alpha}\right)^{\frac{1}{2}}(1 - \tfrac{5}{8}\alpha_n + \ldots), \tag{3.4}$$

$$c_n = 1 + \tfrac{1}{2}\alpha_n - \tfrac{3}{20}\alpha_n^2 + \ldots, \tag{3.5}$$

where

$$G_0 = -2\left(\frac{\alpha_n}{\alpha}\right)\operatorname{sech}\theta. \tag{3.6}$$

Equations (2.12) and (2.6) then yield the known second approximation (see Laitone 1960, for example)

$$\eta = \frac{\alpha_n}{\alpha}\{1 - \tfrac{3}{4}\alpha_n \tanh^2\theta\}\operatorname{sech}^2\theta + O(\alpha^2). \tag{3.7}$$

4. Interaction of two solitary waves

Equation (2.15) is a perturbation of the KdV equation and so we can use the method developed by Byatt-Smith (1987) to study the interaction of two solitary waves. Following the notation of Byatt-Smith (1987), we write (2.15) as

$$u_\tau = \mathrm{X}(u) + \alpha\mathrm{X}^*(u), \tag{4.1}$$

where $$\mathrm{X} = \mathrm{X}^2, \tag{4.2}$$

and $$\mathrm{X}^*(u) = -\tfrac{19}{40}\mathrm{X}^3(u) + \tfrac{3}{16}\{-3uu'' + 5u'u'' + 19u^2u'\}. \tag{4.3}$$

The two-solitary-wave solution of the KdV equation (2.13) is

$$u = -\frac{2\left\{k_1^2 E_1 + k_2^2 E_2 + 2(k_1 - k_2)^2 E_1 E_2 + \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)^2 E_1 E_2 (k_2^2 E_1 + k_1^2 E_2)\right\}}{E^2}, \tag{4.4}$$

where $$E = 1 + E_1 + E_2 + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} E_1 E_2, \tag{4.5}$$

and $$E_i = \exp\{-(\theta_i - \theta_{i_0})\}, \quad i = 1, 2. \tag{4.6}$$

θ_{i_0} are constants and θ_i is given by

$$\theta_i = k_i \xi - \tfrac{1}{2} k_i^3 \tau, \quad i = 1, 2. \tag{4.7}$$

This solution corresponds in (3.7) to an interaction of two solitary waves, of amplitudes $\frac{1}{4}k_1^2$ and $\frac{1}{4}k_2^2$. These two values represent the only two positive eigenvalues of the equation

$$-f'' + uf = -\lambda f. \tag{4.8}$$

The corresponding eigenfunctions are

$$f_i = \frac{E_i^{\frac{1}{2}}\left(1 + \dfrac{(k_i - k_j)}{(k_i + k_j)} E_j\right)}{E}, \quad i = 1, 2, \quad j \neq i. \tag{4.9}$$

The theory of Byatt-Smith (1987) shows that, when u satisfies (4.1) with initial condition (4.4), the rate of change of k_i satisfies the equation

$$\frac{\mathrm{d}k_i}{\mathrm{d}\tau} = -2\alpha \int_{-\infty}^{\infty} f_i^2 \mathrm{X}^*(u), \quad i = 1, 2. \tag{4.10}$$

The higher-order flows are all orthogonal to f_i^2, so that in particular the term $f_i^2 \mathrm{X}^3(u)$ appearing in (4.10) integrates to zero, so that

$$\frac{\mathrm{d}k_1}{\mathrm{d}\tau} = -\frac{3\alpha}{8} \int_{-\infty}^{\infty} f_1^2\{-3uu''' + 5u'u'' + 19u'u^2\} = \frac{\alpha}{8}(9I_1 - 15I_2 - 57I_3). \tag{4.11}$$

These three integrals may be written (see Appendix A) as

$$\begin{aligned} I_1 &= 6I_3 + \tfrac{1}{3}k_2(k_1^2 - 3k_2^2) \int_{-\infty}^{\infty} u(f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}) - k_2^2(k_1^2 - k_2^2) \int_{-\infty}^{\infty} u' f_1^2 f_2^2 \\ &= 6I_3 + \tfrac{1}{3}k_2(k_1^2 - 3k_2^2) I_4 - k_2^2(k_1^2 - k_2^2) I_5, \end{aligned} \tag{4.12}$$

$$I_2 = -\tfrac{1}{2}I_1 - k_2^2 \int_{-\infty}^{\infty} u''\{f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}\} = -\tfrac{1}{2}I_1 - k_2^2 I_6, \tag{4.13}$$

and $$I_3 = -\tfrac{1}{2}k_2^2 \int_{-\infty}^{\infty} u^2\{f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}\} = -\tfrac{1}{2}k_2^2 I_7. \tag{4.14}$$

Equation (4.11) may then be writen as

$$\frac{\mathrm{d}k_1}{\mathrm{d}\tau} = +\frac{k_2^2 \alpha}{16}\{11(k_1^2 - 3k_2^2) I_4 - 33(k_1^2 - k_2^2) I_5 + 30I_6 - 42I_7\}, \tag{4.15}$$

and similarly

$$\frac{\mathrm{d}k_2}{\mathrm{d}\tau} = +\frac{k_1^2\alpha}{16}\{11(k_2^2-3k_1^2)I_4+33(k_1^2-k_2^2)I_5-30I_6+42I_7\}. \tag{4.16}$$

The initial condition (4.4) corresponds to two given solitary waves as $\tau \rightarrow -\infty$. The solution of (4.15) and (4.16) will give the change of k_1 and k_2, and hence the change of amplitude of the two solitary waves, given from (4.6) as

$$\frac{\alpha_i}{\alpha} = \tfrac{1}{4}k_i^2. \tag{4.17}$$

The phase change and resulting radiating wavetrain, or dispersive tail, may also be calculated (see Byatt-Smith (1987) for further details). Since k_1 and k_2 are slowly varying functions of time, we may approximate (4.15) and (4.16) to work out the total change as

$$[k_1] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_1}{\mathrm{d}\tau}\,\mathrm{d}\tau, \tag{4.18}$$

regarding k_1 and k_2 as constant during the integration. Using the equivalence relations relevant to the integrals of I_4, I_5, I_6 and I_7 (see Appendix B) we obtain

$$\begin{aligned}[k_1] &= \frac{\alpha k_2^2}{16}\int_{-\infty}^{\infty}\{11(k_1^2-3k_2^2)I_4-33(k_1^2-k_2^2)I_5+30I_6-42I_7\}\,\mathrm{d}\tau \\ &= \frac{43\alpha}{240}k_2^2(k_1-k_2)\,J(\gamma),\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}J(\gamma) &= k_1\,k_2(k_1-k_2)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{E_1\,E_2(E_1+E_2)}{E^3}\,\mathrm{d}\xi\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cosh\zeta\,\mathrm{d}\zeta\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3},\end{aligned} \tag{4.20}$$

and

$$\gamma = \frac{k_2-k_1}{k_2+k_1}. \tag{4.21}$$

Using the definition of γ we may write

$$[k_1] = -\frac{\frac{43}{120}\alpha k_2^3\,\gamma J(\gamma)}{1+\gamma}. \tag{4.22}$$

The integral J may easily be shown to equal one when $\gamma = 1$, and following Byatt-Smith (1987) we may also obtain the result

$$J(\gamma) \sim -4\gamma\log\gamma \quad \text{as } \gamma \rightarrow 0+. \tag{4.23}$$

For values of γ in the range $0 < \gamma < 1$ the integral is easily computed and is shown in figure 1. Since the integral is positive in this range, (4.22) implies that the smaller solitary wave decreases in amplitude while the larger wave increases in amplitude. Equation (4.23) also implies that these changes tend to zero as $\gamma \rightarrow 0+$. This is the case when the two amplitudes are almost equal. At the other end of the range, where $\gamma \rightarrow 1$, the amplitude of the smaller wave is small and (4.22) implies, as we would expect, that the change in amplitude of the larger wave tends to zero. However if k_1 is the smaller wave and $\alpha_2 = \alpha$ (or $k_2 = 2$), the limiting form for the change in k_1 as $\gamma \rightarrow 1$ is, from (4.22)

$$[k_1] = -\frac{43\alpha}{60}. \tag{4.24}$$

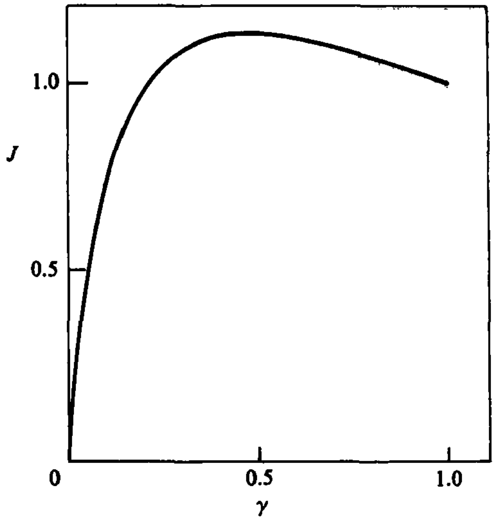

FIGURE 1. Graph of the integral J defined by (4.20) as a function of γ in the range $0 \leqslant \gamma \leqslant 1$.

If this result is valid, then a wave of amplitude $\alpha_1/\alpha < (43\alpha/120)^2$ would disappear completely during its interaction with the larger wave. While this may be the case, the present theory is inadequate to predict such a result. The first reason is that we have taken an approximate solution for (4.14) and (4.15). This may not be accurate enough if k_1 is small. Secondly, the tacit assumption of Byatt-Smith (1987), used in deriving the basic set of equations ((4.10) for example), is that there are no singularities in the spectrum of the operator, $-\mathrm{d}^2/\mathrm{d}\xi^2 + U(\xi)$, appearing on the left-hand side of (4.8). Lastly, the asymptotics used to derive (2.14) may not be valid if the ratio of α_1 to α_2 is not of order one.

To compare the prediction of this theory with the results of Fenton & Rienecker, we need to calculate the change of amplitude of the solitary waves. From (4.17) and (4.22) we can obtain

$$[\alpha_1] \equiv \tfrac{1}{2}\alpha k_1[k_1] = -\tfrac{43}{480}\alpha^2 k_1 k_2^2(k_2 - k_1)\, J(\gamma). \tag{4.25}$$

In terms of the original amplitudes this may be written as

$$[\alpha_1] = -\tfrac{43}{30}\alpha_1^{\frac{1}{2}}\, \alpha_2(\alpha_2^{\frac{1}{2}} - \alpha_1^{\frac{1}{2}})\, J(\gamma), \tag{4.26}$$

and similarly

$$[\alpha_2] = +\tfrac{43}{30}\alpha_2^{\frac{1}{2}}\, \alpha_1(\alpha_2^{\frac{1}{2}} - \alpha_1^{\frac{1}{2}})\, J(\gamma). \tag{4.27}$$

Thus the larger wave has increased in amplitude while the smaller has decreased in amplitude as a result of the interaction. This is at least in qualitative agreement with result of Fenton & Rienecker. However if the values $\alpha_1 = 0.1035$ and $\alpha_2 = 0.3252$ are used in (4.26) and (4.27), we obtain

$$[\alpha_1] = 0.0399, \quad [\alpha_2] = 0.0225. \tag{4.28}$$

This differs substantially from their result and, in comparison with their method for waves travelling in opposite directions, it appears that their result has changes of magnitude of order α^3, compared with the above result which is of order α^2. Also, (4.26) and (4.27) show that $|[\alpha_1]/[\alpha_2]|$, which equals $(\alpha_2/\alpha_1)^{\frac{1}{2}}$, is always greater than one. This is consistent with the fact that the energy of the two waves is *not* increased as result of the interaction. Since the energy of a solitary wave of small amplitude α is proportional to $\alpha^{\frac{3}{2}}$, it follows that

$$\frac{[E]}{E} = 0 \quad \text{to order } \alpha, \tag{4.29}$$

a result that can be obtained directly from (4.15) and (4.16), which can be integrated to give the conservation law

$$k_1^3 + k_2^3 = \text{const.} \tag{4.30}$$

Since the energy of each wave cannot be altered without the production of a dispersive tail, the above conservation law implies that the order of magnitude of the dispersive tail is $O(\alpha^3)$, rather than $O(\alpha^2)$, which we would expect from an order-α^2 perturbation that (4.1) represents.

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Appendix A

In this Appendix we reduce the three integrals appearing in (4.11) to the expressions given in (4.12)–(4.14).

We shall make use of the results (see Byatt-Smith 1987)

$$\mathrm{X}(f_i^2) = -\tfrac{1}{2}k_i^2 f_i^{2\prime} \quad (i = 1, 2), \tag{A 1}$$

$$\mathrm{X}(u) = -\tfrac{1}{2}u''' + 3uu', \tag{A 2}$$

and

$$u = -2(k_1^2 f_1^2 + k_2^2 f_2^2). \tag{A 3}$$

Using (A 3) we may obtain, using integration by parts,

$$\int_{-\infty}^{\infty} u(f_1^2 f_2^{2\prime} - f_1^{2\prime} f_2^2) = 6 \int_{-\infty}^{\infty} f_1^2 f_2^2 (k_1^2 f_1^{2\prime} - k_2^2 f_2^{2\prime}). \tag{A 4}$$

Then we may express I_1 as

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} f_1^2 uu''' \\ &= 2 \int_{-\infty}^{\infty} f_1^2 u(-\mathrm{X}(u) + 3uu') \\ &= 4 \int_{-\infty}^{\infty} f_1^2 u\{k_1^2\, \mathrm{X}(f_1^2) + k_2^2\, \mathrm{X}(f_2^2)\} + 6 \int_{-\infty}^{\infty} f_1^2 u^2 u' \\ &= 4 \int_{-\infty}^{\infty} f_1^2 (k_1^2 f_1^2 + k_2^2 f_2^2)\,(k_1^4 f_1^{2\prime} + k_2^4 f_2^{2\prime}) + 6I_3 \\ &= 4 \int_{-\infty}^{\infty} \{f_1^2 f_2^2\, k_1^4\, k_2^2 f_1^{2\prime} + k_1^2\, k_2^4 f_1^4 f_2^{2\prime} + k_2^6 f_2^2 f_1^2 f_2^{2\prime}\} + 6I_3 \end{aligned}$$

$$= 4k_2^2 \int_{-\infty}^{\infty} f_1^2 f_2^2 \{(k_1^4 - 2k_1^2 k_2^2) f_1^{2\prime} + k_2^4 f_2^{2\prime}\} + 6I_3$$

$$= 4k_2^2 \int_{-\infty}^{\infty} f_1^2 f_2^2 \{\tfrac{1}{2}(k_1^2 - k_2^2)(k_1^2 f_1^{2\prime} + k_2^2 f_2^{2\prime}) + \tfrac{1}{2}(k_1^2 - 3k_2^2)(k_1^2 f_1^{2\prime} - k_2^2 f_2^{2\prime})\} + 6I_3$$

$$= -k_2^2(k_1^2 - k_2^2) \int_{-\infty}^{\infty} f_1^2 f_2^2 u' + \tfrac{1}{3}(k_1^2 - 3k_2^2) k_2^2 \int_{-\infty}^{\infty} u(f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}) + 6I_3. \tag{A 5}$$

If we now take the combination $I_2 + \frac{1}{2}I_1$, we may show that

$$I_2 + \tfrac{1}{2}I_1 \equiv \int_{-\infty}^{\infty} f_1^2 u'u'' + \tfrac{1}{2}\int f_1^2 uu'''$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (f_1^2 u'u'' - f_1^{2\prime} uu'')$$

$$= -\int_{-\infty}^{\infty} \{f_1^2(k_1^2 f_1^{2\prime} + k_2^2 f_2^{2\prime}) - f_1^{2\prime}(k_1^2 f_1^{2\prime} + k_2^2 f_1^{2\prime})\} u''$$

$$= -k_2^2 \int_{-\infty}^{\infty} (f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}) u''. \tag{A 6}$$

Hence
$$I_2 = -\tfrac{1}{2}I_1 - k_2^2 \int_{-\infty}^{\infty} (f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}) u''. \tag{A 7}$$

Finally one integration by parts shows that

$$I_3 = \int_{-\infty}^{\infty} f_1^2 u^2 u'$$

$$= -\int_{-\infty}^{\infty} f_1^{2\prime} u^3 - 2\int_{-\infty}^{\infty} f_1^2 u^2 u'. \tag{A 8}$$

Thus we may write

$$I_3 = \frac{1}{4}\int_{-\infty}^{\infty} (f_1^2 u^2 u' - f_1^{2\prime} u^3)$$

$$= -\tfrac{1}{4}k_2^2 \int u^2 \{f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime}\}. \tag{A 9}$$

Appendix B

In this Appendix we show that integrals I_4, I_5, I_6 and I_7 defined by (4.12)–(4.14) may be reduced to one integral.

Equation (4.4) may be rewritten as

$$u = -2\left(\frac{E''}{E} - \frac{E'^2}{E^2}\right) \tag{B 1}$$

and (4.9) may be used to derive the expressions (see Byatt-Smith 1987 for more details)

$$f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime} = \frac{k_1 - k_2}{k_1 + k_2} \frac{E_1 E_2}{E_3} \{2E' + (k_1 + k_2) E\}, \tag{B 2}$$

and
$$f_1^2 f_2^2 = \frac{E_1 E_2}{E^4} \left\{\frac{2E'}{k_1 + k_2} + E\right\}^2. \tag{B 3}$$

We now introduce the concept of equivalence and write

$$G_1(\tau) \cong G_2(\tau) \tag{B 4}$$

if

$$\int_{-\infty}^{\infty} G_1(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} G_2(\tau)\,\mathrm{d}\tau. \tag{B 5}$$

Byatt-Smith (1987) effectively proves that

$$\int_{-\infty}^{\infty} I_4\,\mathrm{d}\tau \equiv \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} u(f_1^2 f_2^{2\prime} - f_2^2 f_1^{2\prime})\right\}\mathrm{d}\tau = 0, \tag{B 6}$$

that is,

$$I_4 \cong 0. \tag{B 7}$$

This may be rewritten to give

$$\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^3} E_1 \cong \int_{-\infty}^{\infty} \frac{E_1 E_2}{E^3} E_2. \tag{B 8}$$

Similarly we may show that (see for example (B 22))

$$\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4} E_1 \cong \int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4} E_2. \tag{B 9}$$

Using (B 1) and (B 3) in the definition of I_5 we obtain

$$I_5 = \int_{-\infty}^{\infty} u' f_1^2 f_2^2 = -\frac{2}{(k_1+k_2)^2}\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^7}(E'''E^2 - 3E''E'E + 2E'^3)(2E' + (k_1+k_2)E)^2. \tag{B 10}$$

The integral is now expanded in inverse powers of E to obtain

$$\begin{aligned} I_5 = -\frac{2}{(k_1+k_2)^2}\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^7}\{&E^4 E'''(k_1+k_2)^2 + E^3(4E'''E'(k_1+k_2) \\ &-3E''E'(k_1+k_2)^2) + E^2(4E''E'^2 - 12(k_1+k_2)E''E'^2 + 2(k_1+k_2)E'^3) \\ &-E(12E''E'^3 - 8(k_1+k_2)E'^4) + 8E'^5\}. \end{aligned} \tag{B 11}$$

The last term may be integrated by parts to give

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{8E'^5 E_1 E_2}{E^7} &= \left[-\frac{4}{3}E'^4 \frac{E_1 E_2}{E^6}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (16E''E'^3 - 4(k_1+k_2)E'^4)\frac{E_1 E_2}{3E^6} \\ &= \int_{-\infty}^{\infty} (16E''E'^3 - 4(k_1+k_2)E'^4)\frac{E_1 E_2}{3E^6}. \end{aligned} \tag{B 12}$$

This process is repeated three times to give

$$\begin{aligned} I_5 = &-\frac{2}{(k_1+k_2)^2}\int_{-\infty}^{\infty} \frac{E_1 E_2}{18E^3}\{4(k_1+k_2)E^{\mathrm{iv}} + 8(k_1+k_2)^2 E''' + 5(k_1+k_2)^3 E'' + (k_1+k_2)^4 E'\} \\ &-\frac{2}{(k_1+k_2)^2}\int_{-\infty}^{\infty} \{\tfrac{2}{3}(E'E^{\mathrm{iv}} - E'''E'') + (k_1+k_2)(E'''E' - E''^2)\}\frac{E_1 E_2}{E^4}. \end{aligned} \tag{B 13}$$

The second integral appears because the two terms $E'''E''E_1E_2E^{-4}$ and $E''^2E_1E_2E^{-4}$ cannot be integrated by parts. The two additional terms have been introduced to reduce the integral to a simple form.

Using the equivalence relation (B 8) the first integral becomes

$$\int_{-\infty}^{\infty} \frac{E_1 E_2}{18E^3}\{4(k_1+k_2)E^{\mathrm{iv}}+8(k_1+k_2)^2E'''+5(k_1+k_2)^3E''+(k_1+k_2)^4E'\}$$

$$\cong \int_{-\infty}^{\infty} \frac{E_1 E_2}{18E^3}(E_1+E_2)\{2(k_1+k_2)(k_1^4+k_2^4)-4(k_1+k_2)^2(k_1^3+k_2^3)$$

$$+\tfrac{5}{2}(k_1+k_2)^3(k_1^2+k_2^2)-\tfrac{1}{2}(k_1+k_2)^5\}$$

$$= k_1 k_2(k_1+k_2)(k_1-k_2)^2 \int_{-\infty}^{\infty} \frac{E_1 E_2(E_1+E_2)}{E^3}. \qquad \text{(B 14)}$$

Using (4.5) we may reduce the second integral of (B 13) to

$$\int_{-\infty}^{\infty} \{\tfrac{2}{3}(E'E^{\mathrm{iv}}-E'''E'')+(k_1+k_2)(E'''E'-E''^2)\}\frac{E_1 E_2}{E^4}$$

$$= \tfrac{1}{3}k_1 k_2(k_1-k_2)^2(k_1+k_2)\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4}\left\{1+\frac{k_1-k_2}{k_1+k_2}(k_1 E_2-k_2 E_1)\right\}$$

$$\cong \tfrac{1}{3}k_1 k_2(k_1-k_2)(k_1+k_2)\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4}\left\{1+\frac{1}{2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2(E_1+E_2)\right\}, \qquad \text{(B 15)}$$

by virtue of (B 9).

The next step is to show that the integrals in (B 14) and (B 15) are related by the equivalence relation

$$k_1 k_2(k_1-k_2)^2 \int_{-\infty}^{\infty} \frac{E_1 E_2}{E^3}(E_1+E_2)$$

$$\cong 6k_1 k_2(k_1-k_2)\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4}\left\{1+\frac{1}{2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2(E_1+E_2)\right\}. \qquad \text{(B 16)}$$

We first transform the equation by integrating with respect to time (τ) so that the left-hand side becomes

$$\int_{-\infty}^{\infty}\left\{k_1 k_2(k_1-k_2)^2\int_{-\infty}^{\infty}\frac{E_1 E_2(E_1+E_2)}{E^3}\,\mathrm{d}\xi\right\}\mathrm{d}\tau$$

$$= \tfrac{1}{4}k_1 k_2(k_1-k_2)^2$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cosh\{\frac{1}{2}(k_2-k_1)\xi-\frac{1}{4}(k_2^3-k_1^3)\tau\}\,\mathrm{d}\xi\,\mathrm{d}\tau}{\{\mathrm{e}^{-\theta}\cosh\frac{1}{2}(k_1+k_2)\xi-\frac{1}{4}(k_1^3+k_2^3)\tau+\theta)+\cosh(\frac{1}{2}(k_2-k_1)\xi-\frac{1}{4}(k_2^3-k_1^3)\tau)\}^3}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cosh\zeta\,\mathrm{d}\zeta\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3}, \qquad \text{(B 17)}$$

where

$$\gamma = \mathrm{e}^{-\theta} = \frac{k_2-k_1}{k_2+k_1}. \qquad \text{(B 18)}$$

The last line is obtained by using the transformations

$$\tfrac{1}{2}(k_2-k_1)\xi-\tfrac{1}{4}(k_2^3-k_1^3)\tau = \zeta \qquad \text{(B 19)}$$

and

$$\tfrac{1}{2}(k_2+k_1)\xi-\tfrac{1}{4}(k_2^3+k_1^3)\tau+\theta = \frac{\rho}{\gamma}. \qquad \text{(B 20)}$$

The same set of transformations applied to the right-hand side of (B 16) gives

$$6k_1 k_2(k_1-k_2)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4}(1+\tfrac{1}{2}\mathrm{e}^{-2\theta}(E_1+E_2))\,\mathrm{d}\xi\,\mathrm{d}\tau$$

$$=\frac{3}{2}\int_{-\infty}^{\infty}\int_{-\infty} \frac{(1+\gamma\,\mathrm{e}^{-\rho/\gamma}\cosh\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}$$

$$=\frac{3}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(1+\gamma\cosh(\rho/\gamma)\cosh\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}. \quad \text{(B 21)}$$

To obtain the last line we have used the simple result that

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\rho/\gamma}\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{+\rho/\gamma}\,\mathrm{d}\rho}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}. \quad \text{(B 22)}$$

We note here that a similar transformation of the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{E_1^2 E_2^2}{E^4}(E_1-E_2)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

shows that it is equal to zero, which gives the result of (B 9).

Finally, an integration by parts gives

$$\int_{-\infty}^{\infty} \frac{\cosh\zeta\,\mathrm{d}\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3} = \left[\frac{\sinh\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{3\sinh^2\zeta\,\mathrm{d}\tau}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}$$

$$=3\int_{-\infty}^{\infty} \frac{\{\cosh\zeta(\cosh\zeta+\gamma\cosh(\rho/\gamma))-1-\gamma\cosh\zeta\cosh(\rho/\gamma)\}\,\mathrm{d}\tau}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}$$

$$=3\int_{-\infty}^{\infty} \frac{\cosh\zeta\,\mathrm{d}\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3} - 3\int_{-\infty}^{\infty} \frac{\{1+\gamma\cosh\zeta\cosh\rho/\gamma\}\,\mathrm{d}\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}. \quad \text{(B 23)}$$

Hence by rearranging this equation we obtain

$$\int_{-\infty}^{\infty} \frac{\cosh\zeta\,\mathrm{d}\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^3} = \frac{3}{2}\int_{-\infty}^{\infty} \frac{\{1+\gamma\cosh\zeta\cosh(\rho/\gamma)\}\,\mathrm{d}\zeta}{(\gamma\cosh(\rho/\gamma)+\cosh\zeta)^4}. \quad \text{(B 24)}$$

This equation together with expression (B 17) and (B 21) establishes the equivalence relation of (B 16). Hence

$$I_5 \cong -\tfrac{2}{9}k_1 k_2 \frac{(k_1-k_2)^2}{k_1+k_2}\int_{-\infty}^{\infty} \frac{E_1 E_2(E_1+E_2)}{E^3}. \quad \text{(B 25)}$$

Using the same procedure on the remaining integrals we obtain the results

$$I_6 = -2\frac{k_1-k_2}{k_1+k_2}\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^7}(E^3E^{\mathrm{iv}}-4(E'''E'+3E''^2)E^2$$

$$+12E''E'^2E-6E'^4)(2E'+(k_1+k_2)E)$$

$$\cong -\tfrac{1}{3}k_1 k_2(k_1-k_2)^3\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^3}(E_1+E_2), \quad \text{(B 26)}$$

and

$$I_7 = 4\frac{(k_1-k_2)}{k_1+k_2}\int_{-\infty}^{\infty} \frac{E_1 E_2}{E^7}(EE''-E'^2)^2(2E'+(k_1+k_2)E)$$

$$\cong -\tfrac{1}{5}k_1 k_2(k_1-k_2)^3\int_{-\infty}^{\infty} \frac{E_1 E_2(E_1+E_2)}{E^3}. \quad \text{(B 27)}$$

REFERENCES

Boussinesq, M. J. 1872 *J. Math. Pures Appl.* (2), 17, 55.

Byatt-Smith, J. G. B. 1987 *J. Fluid Mech.* **182**, 467.

Ertekin, R. C. 1984 Soliton generation by moving disturbances in shallow water: theory, computation and experiment. Ph.D. thesis, University of California, Berkeley.

Ertekin, R. C., Webster, W. C. & Wehausen, J. V. 1984 *Proc. 15th Symp. Naval Hydrodynamics, Hamburg*, pp. 347–361. Washington D.C: National Academy Press.

Ertekin, R. E., Webster, W. C. & Wehausen, J. V. 1986 *J. Fluid Mech.* **169**, 275.

Fenton, J. D. & Rienecker, M. M. 1982 *J. Fluid Mech.* **118**, 411.

Huang, de-B., Sibul, O. J., Webster, W. C., Wehausen, J. V., Wu, de-M. & Wu, T. Y. 1982 Ships moving in the transcritical range. In *Proc. Conf. on Behavior of Ships in Restricted Waters*, vol. 2. Varna.

Johnson, R. S. 1982 *J. Fluid Mech.* **120**, 49.

Karpman, V. I. & Maslov, E. M. 1977*a* *Phys. Lett.* **60A**, 307.

Karpman, V. I. & Maslov, E. M. 1977*b* *Phys. Lett.* **61A**, 355.

Karpman, V. I. & Maslov, E. M. 1977*c* *Sov. Phys., J. Exp. Theor. Phys.* **73**, 537.

Kaup, D. J. & Newell, A. C. 1978 *Proc. R. Soc. Lond.* A **361**, 413.

Keener, J. & McLaughlin, D. W. 1977 *Phys. Rev.* **A16**, 777.

Korteweg, D. J. & de Vries, G. 1895 *Phil. Mag.* **39**, 422.

Laitone, E. V. 1960 *J. Fluid Mech.* **9**, 430.

Longuet-Higgins, M. S. & Fenton, J. D. 1974 On the mass momentum, energy and circulation of a solitary wave II. *Proc. R. Soc. Lond.* A **340**, 471.

McKean, H. P. & van Morbeke, P. 1975 *Invent. Math.* **30**, 217.

Miles, J. W. 1977 *J. Fluid Mech.* **79**, 157.

Scott-Russell, R. 1844 In *Rep. of the 14th Meeting of the British Association for the Advancement of Science, York*, pp. 311–390. London: John Murray.

Su, C. H. & Mirie, R. M. 1980 *J. Fluid Mech.* **98**, 509.

Su, C. H. & Mirie, R. M. 1982 *J. Fluid Mech.* **115**, 475.